Assignment 5

Hand in Ex 3.2, no. 3, Supp. Ex no. 4, 7 by March 7.

Exercise 3.2 no. 1, 3, 4.

Supplementary Exercises

The space $R[a,b]$ consisting of all Riemann integrable functions is endowed with the inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx \ .$$

Set $\|f\| = \sqrt{\langle f, f \rangle}$.

1. Let $\{v_j\}_{j=1}^N, N \leq \infty$, be an orthonormal set in V. Prove Bessel's inequality

$$\sum_{j=1}^{N} \langle v, v_j \rangle^2 \leq \|v\|^2 \ .$$

2. Establish Cauchy-Schwarz inequality $|\langle u, v \rangle| \leq \|u\|\|v\|$ in an inner product space and then use it to prove the triangle inequality

$$\|u + v\| \leq \|u\| + \|v\| \ .$$

   Do it in both real and complex cases.

3. Show that for $f \in R[a,b]$,

$$\|f\| \leq (b-a)\|f\|_\infty \ ,$$

   where $\|f\|_\infty = \sup_x |f(x)|$ is the sup-norm of f. Then use it to show f_n tends to f in L^2-norm if f_n tends to f uniformly, but the converse is not true.

4. The sequence $\{f_n\}$ is called pointwisely convergent to f if $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in [a,b]$. Construct (a) a pointwisely convergent but not L^2-convergent sequence, and (b) an L^2-norm convergent but not pointwisely convergent sequence. You may work on $[0,1]$.

5. Let W be a subspace in V and $\{w_1, \cdots, w_n\}$ be an orthonormal basis of W. Suppose that $w_1 \in W$ satisfies $\langle u - w_1, w \rangle = 0$ for all $w \in W$. Show that w_1 is the orthogonal projection of u on W.

6. Verify that the orthogonal projection of f on the subspace E_n (see page 3, Notes 4) is equal to $S_n f$, the n-th partial sum of the Fourier series of f.

7. Establish the identity

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \ ,$$

   by looking at the Parseval's identity for the function $f(x) = x$.